# Fast, uniform scalar multiplication for genus 2 Jacobians with fast Kummers *and* $\mu$Kummer: efficient hyperelliptic signatures and key exchange on microcontrollers

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# *Genus 2 cryptography*

Genus 2 is an alternative to elliptic curves (genus 1) as a source of groups for DLP-based cryptosystems.

So far: best DLP algorithm is Pollard $\rho$

$\Longrightarrow$ same DLP hardness per bit as elliptic curves

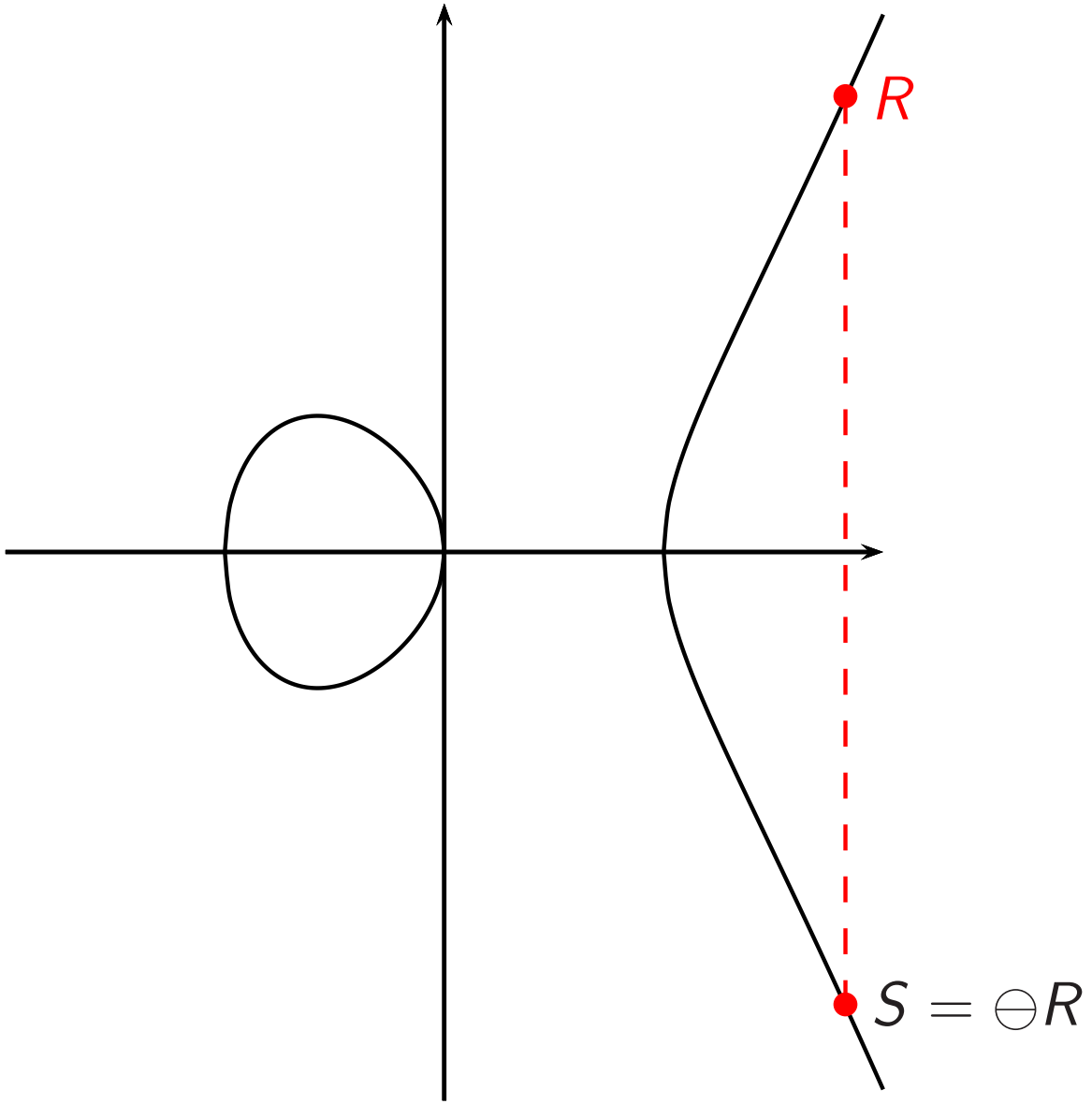
$\Longrightarrow$ same size keys (optimally small).

*Why bother?*

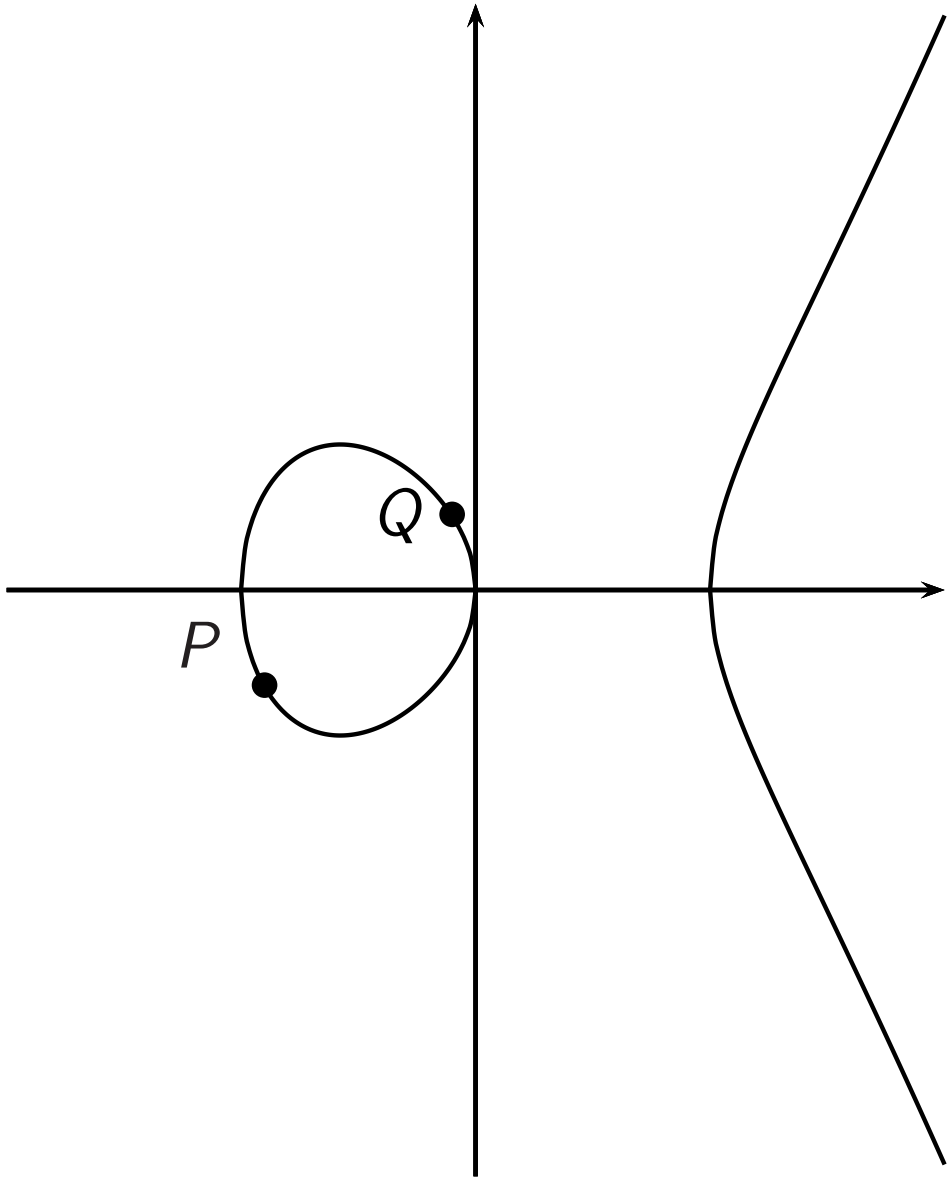
Sometimes genus 2 can offer better performance.

Plus, it's much more fun.

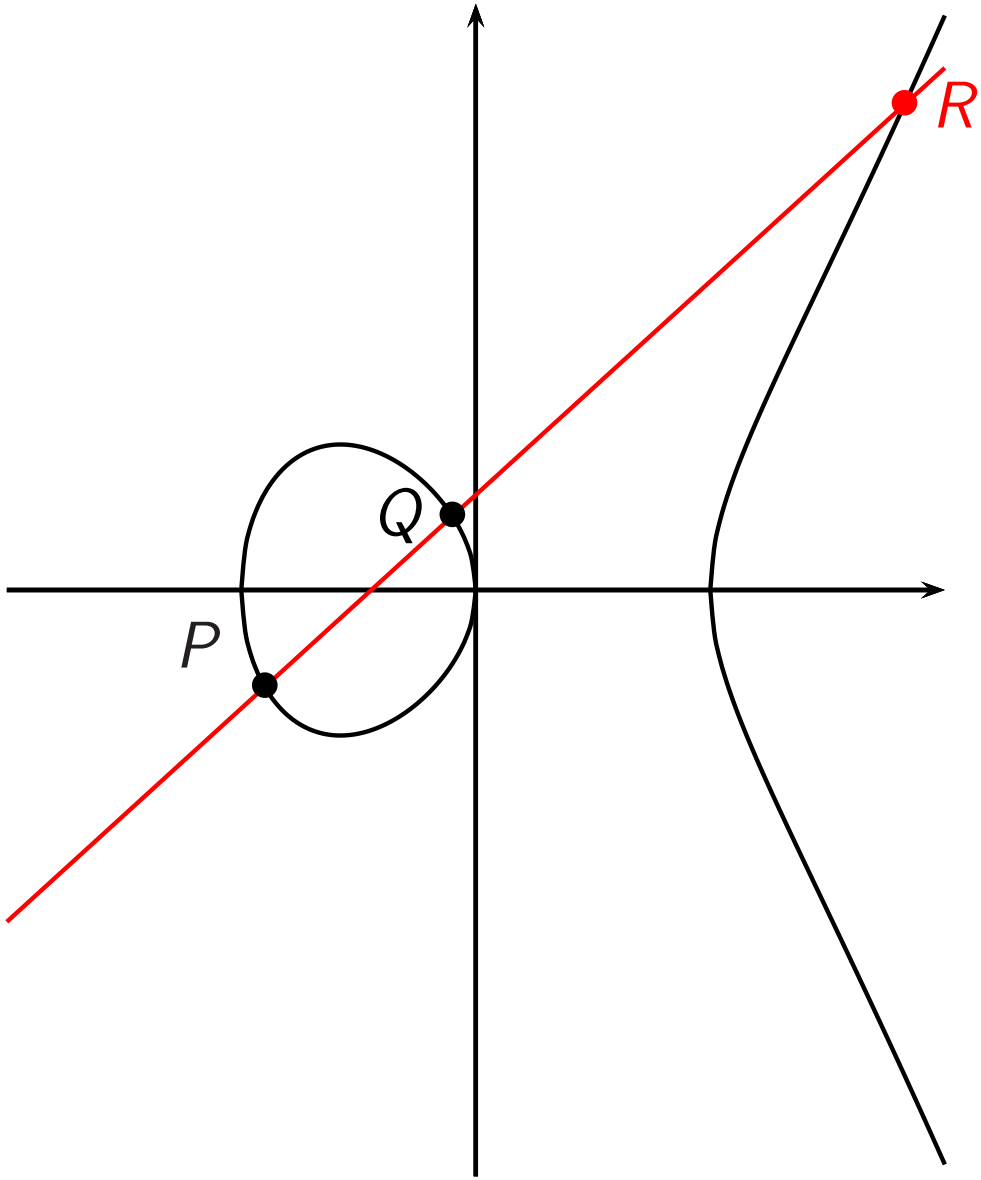
# Elliptic curve negation: $\ominus R = S$



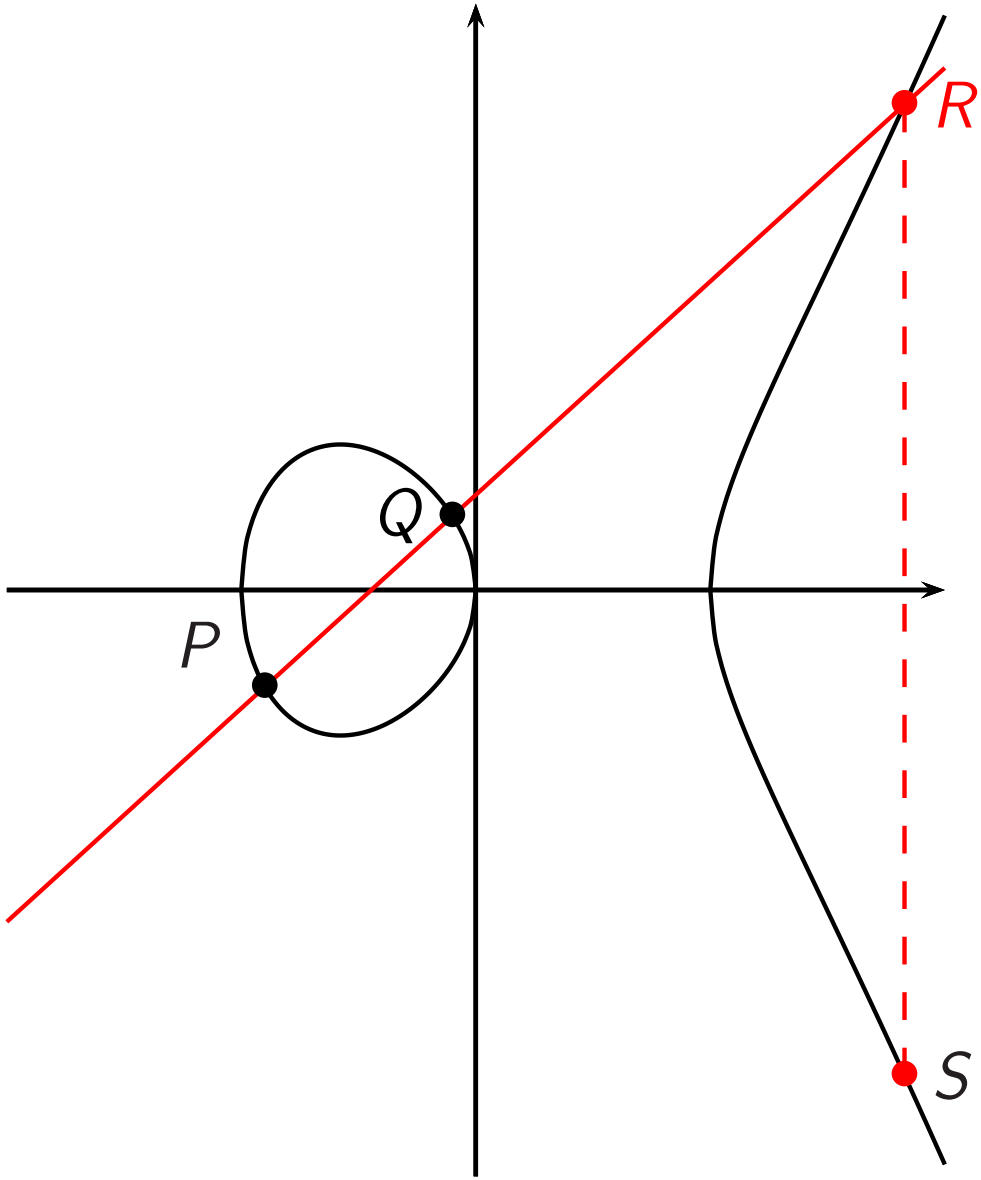
# *Elliptic curve addition: $P \oplus Q = ?$*



# *Elliptic curve addition:* $P \oplus Q \oplus R = 0$

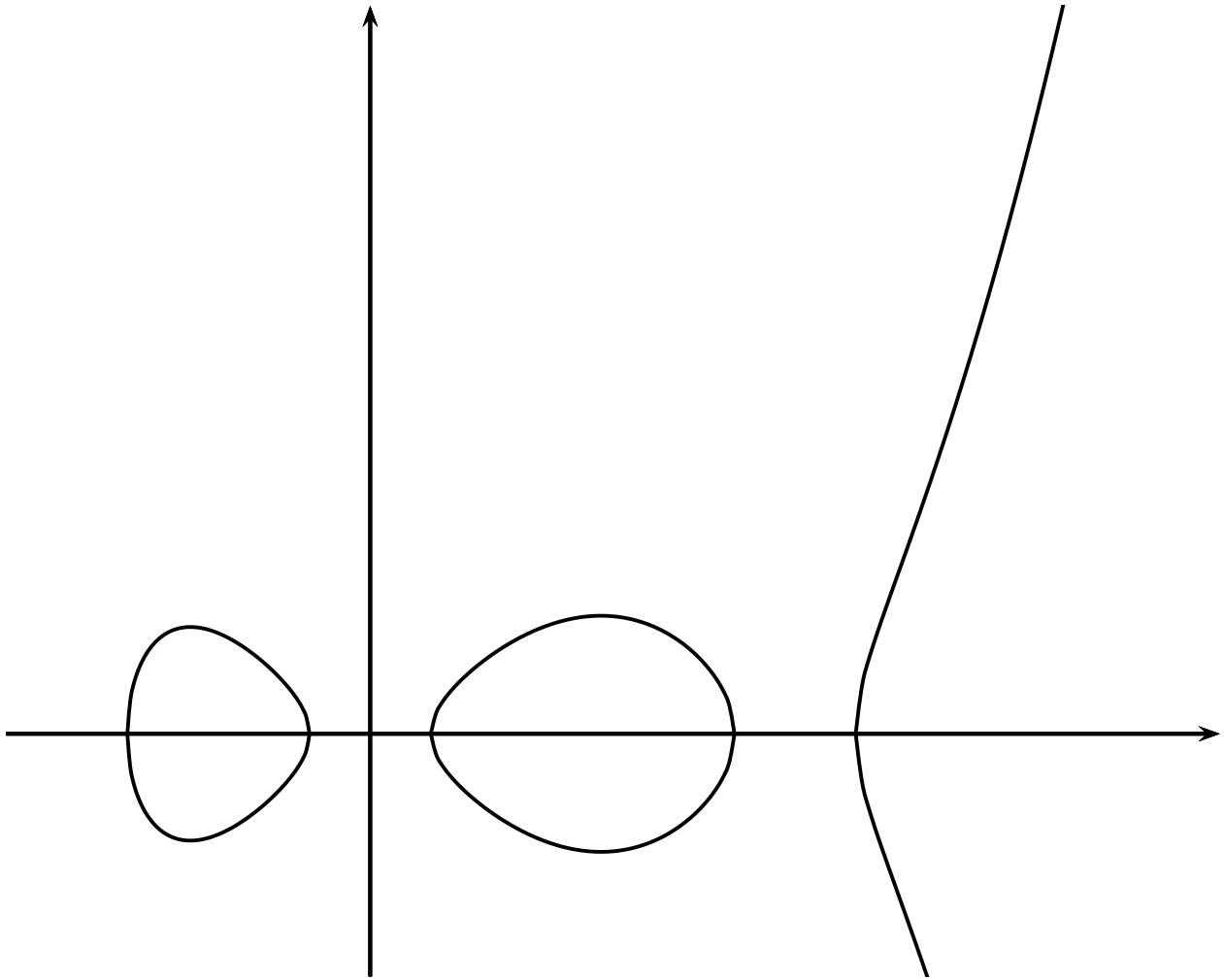


# *Elliptic curve addition: $P \oplus Q = \ominus R = S$*



# Genus 2 curves

$\mathcal{C} : y^2 = f(x)$ with $f \in \mathbb{F}_p[x]$ degree 5 or 6 and squarefree



# *Making groups from genus 2 curves*

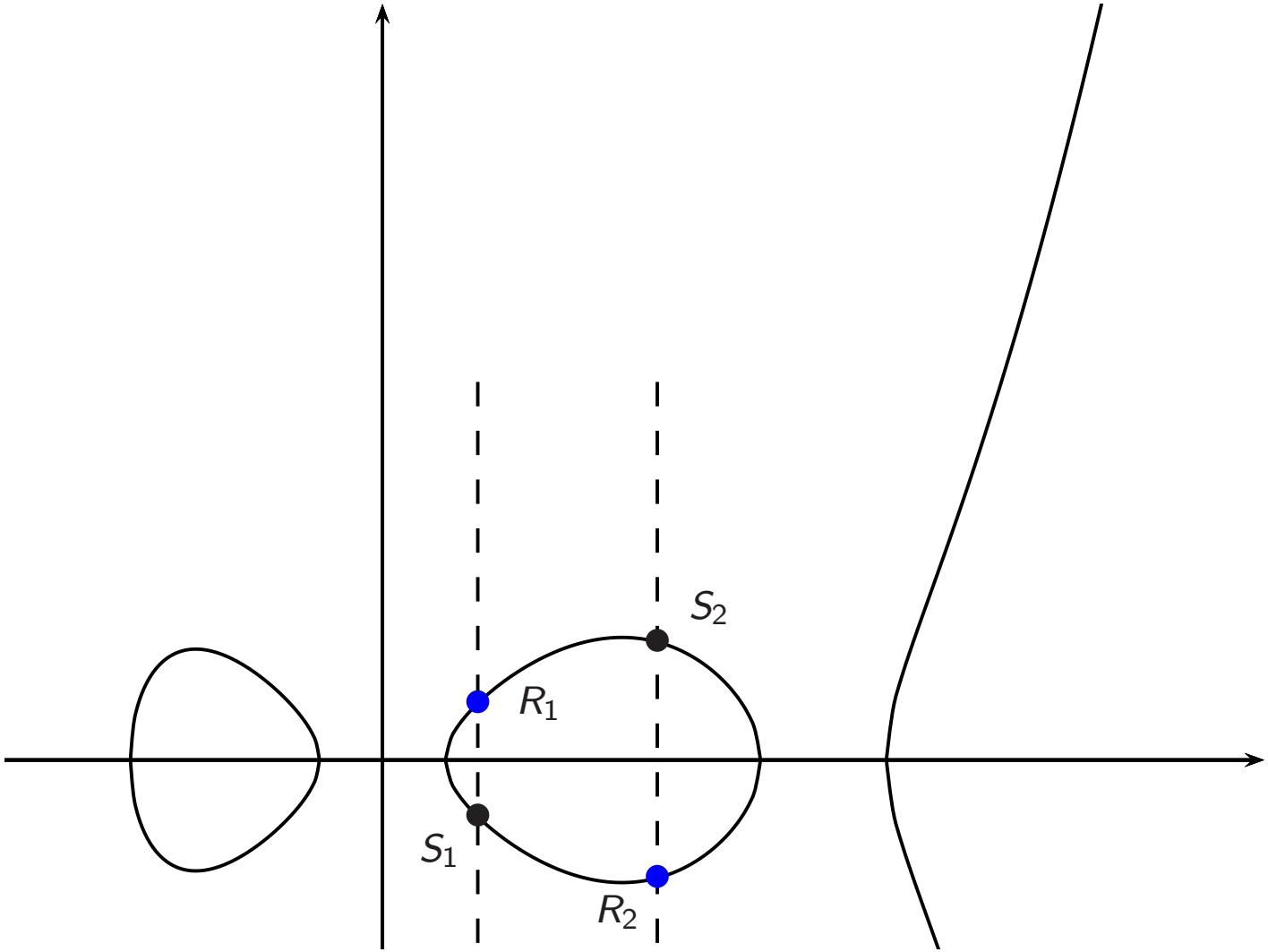
*Jacobian*: algebraic group $\mathcal{J}_{\mathcal{C}} \sim \mathcal{C}^{(2)}$:
Elements are pairs of points on $\mathcal{C} : y^2 = f(x)$,
with all pairs $\{(x, y), (x, -y)\}$ "blown down" to 0.

Negation $\ominus : \{(x_1, y_1), (x_2, y_2)\} \mapsto \{(x_1, -y_1), (x_2, -y_2)\}$

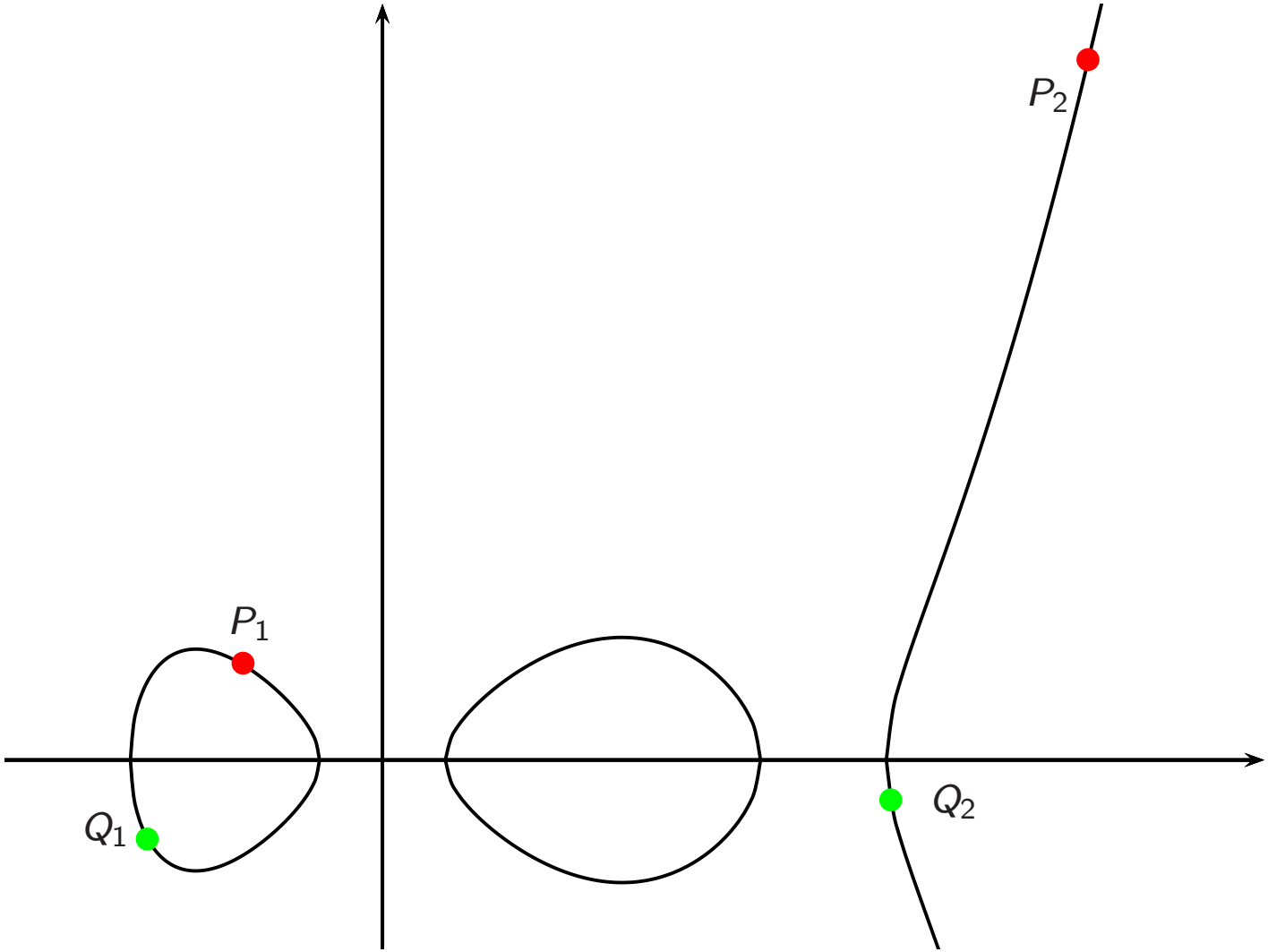
Group law on $\mathcal{J}_{\mathcal{C}}$ induced by
$\{P_1, P_2\} \oplus \{Q_1, Q_2\} \oplus \{R_1, R_2\} = 0$
whenever $P_1, P_2, Q_1, Q_2, R_1, R_2$ are
the intersection of $\mathcal{C}$ with some cubic $y = g(x)$.

*Why? Any 4 points in the plane determine a cubic $y = g(x)$, which must intersect $\mathcal{C} : y^2 = f(x)$ in 6 points because $g(x)^2 = f(x)$ has 6 solutions.*

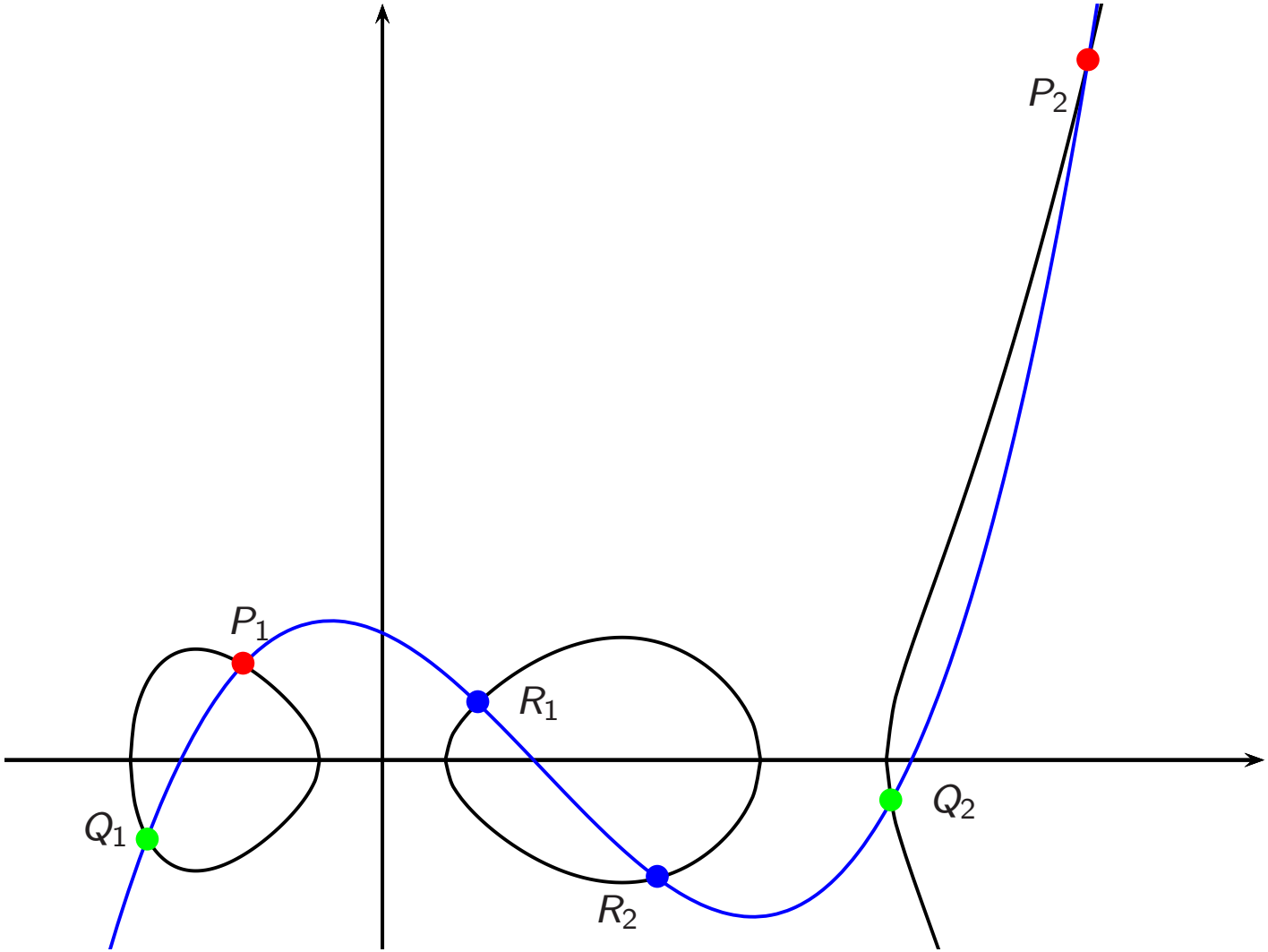
## *Genus 2 group law: negation* $\ominus\{R_1, R_2\} = \{S_1, S_2\}$



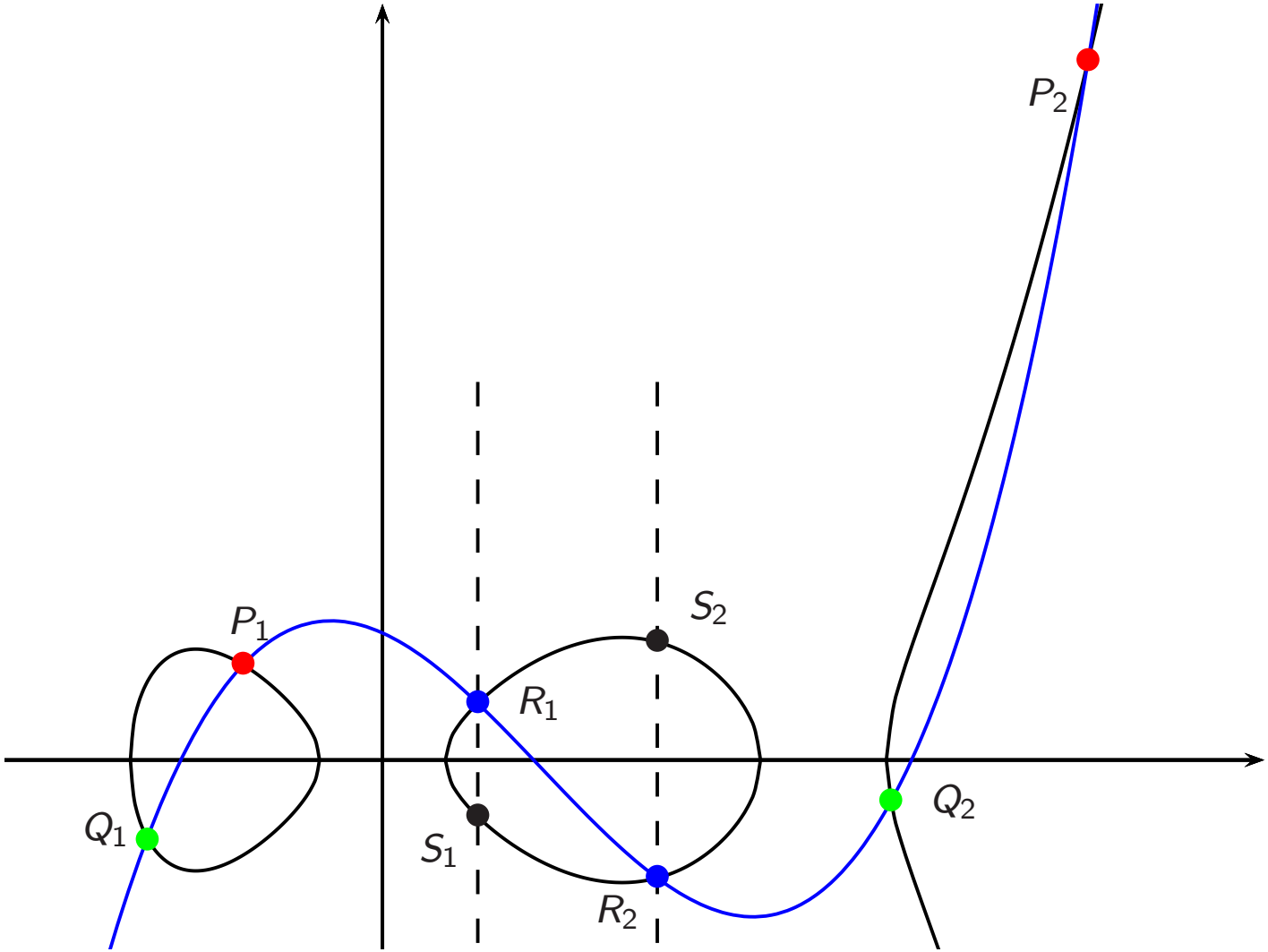
# *Genus 2 group law:* $\{P_1, P_2\} \oplus \{Q_1, Q_2\} = ?$



## *Genus 2 group law:* $\{P_1, P_2\} \oplus \{Q_1, Q_2\} \oplus \{R_1, R_2\} = 0$



## *Genus 2 group law:* $\{P_1, P_2\} \oplus \{Q_1, Q_2\} = \ominus\{R_1, R_2\} = \{S_1, S_2\}$



We (still) want to implement basic cryptosystems based on Discrete Log and Diffie–Hellman problems in $\mathcal{J}_{\mathcal{C}}(\mathbb{F}_p)$

Protocols like Diffie–Hellman key exchange, Schnorr signatures, (EC)DSA signatures:

We *occasionally* need to compute $P \oplus Q$.

We *mostly* need to compute $[m]P$.

Side channel safety $\Longrightarrow$ scalar multiplication must be *uniform* and *constant-time* when the scalar $m$ is secret.

# Schnorr signatures: signing

Fixed public base point $P$ in a $2t$-bit group; $N = \#\langle P \rangle$.
Keys: private $x$, public $Q = [x]P$.

To sign a message $m$ with the key pair $(Q, x)$:

**Algorithm 1** Schnorr signature signing operation

1: **function** $\textsc{Sign}(m \in \{0,1\}^*, x \in \mathbb{Z}/N\mathbb{Z})$
2: $r \leftarrow$ **random**$(\mathbb{Z}/N\mathbb{Z})$
3: $R \leftarrow [r]P$ ▷ mult. public point by secret scalar
4: $e \leftarrow H(m||R)$ ▷ $H$: hash function, $t$ bit output
5: $s \leftarrow r - ex \pmod N$ ▷ (so $[s]P \oplus [e]Q = R$)
6: **return** $(s, e) \in (\mathbb{Z}/N\mathbb{Z}) \times [0..2^t)$ ▷ $3t$ bits of storage
7: **end function**

$\Longrightarrow$ *signatures at the 128-bit security level:*
*Elliptic and genus 2 Schnorr require 384 bits, RSA requires 3072.*

# Schnorr signatures: verification

To verify a claimed signature $(m, e)$ on a message $m$ against a public key $Q$,

**Algorithm 2** Schnorr signature verification

1: **function** VERIFY$((s, e), m, Q)$
2: $\quad R' \leftarrow [s]P \oplus [e]Q$ ▷ mult. public points, scalars
3: $\quad e' \leftarrow H(m||R')$
4: $\quad$ **return** $e' = e$
5: **end function**

DLP hardness + hash preimage hardness
gives $t$ bits of authenticity, integrity, and non-repudiability.

## *...So you want to instantiate a DLP/DHP-based protocol*

Smallest key size for a given security level:
use an *elliptic curve* or a *genus 2 Jacobian*.

*For signatures and encryption*:

Elliptic: Edwards curves (eg. Ed25519), NIST curves, etc.

Genus 2: Jacobian surfaces.

Scalar mult: *Uniform* genus 2 is much slower than elliptic curves.

*For Diffie–Hellman*:

Elliptic: $x$-lines of Montgomery curves (eg. Curve25519)

Genus 2: Kummer surfaces (Jacobians modulo $\pm 1$).

Scalar mult: *Uniform* genus 2 can be faster than elliptic curves.
*E.g.: Bos–Costello–Hisil–Lauter (2013+14),*
*Bernstein–Chuengsatiansup–Lange–Schwabe (Asiacrypt 2014)*

# *Why is uniform genus 2 tricky?*

Elements $\{P_1, P_2\}$: separate, *incompatible* representations for cases where one or both of the $P_i$ are at infinity.

Group law $\{P_1, P_2\} \oplus \{Q_1, Q_2\} = \{S_1, S_2\}$: branch-tacular, separate special cases for $P_i$, $Q_i$ at infinity, for $P_i = P_j$, for $P_i = Q_j$, for $\{P_1, P_2\} = \{Q_1, Q_2\}$, ...

These special cases are never implemented in "record-breaking" genus 2 implementations, but they're easy to attack in practice.

For elliptic curves, we can always sweep the special cases under a convenient line to get a uniform group law, but in genus 2 this is much harder; *protection kills performance*.

# *Why is Diffie–Hellman different?*

Now you know why genus 2 Jacobians are painful candidates for cryptographic groups in practice.

So why is genus 2 fast and safe for Diffie–Hellman?

Because DH *doesn't need a group law*,
just scalar multiplication;
so we can "drop signs" and work modulo $\ominus$.

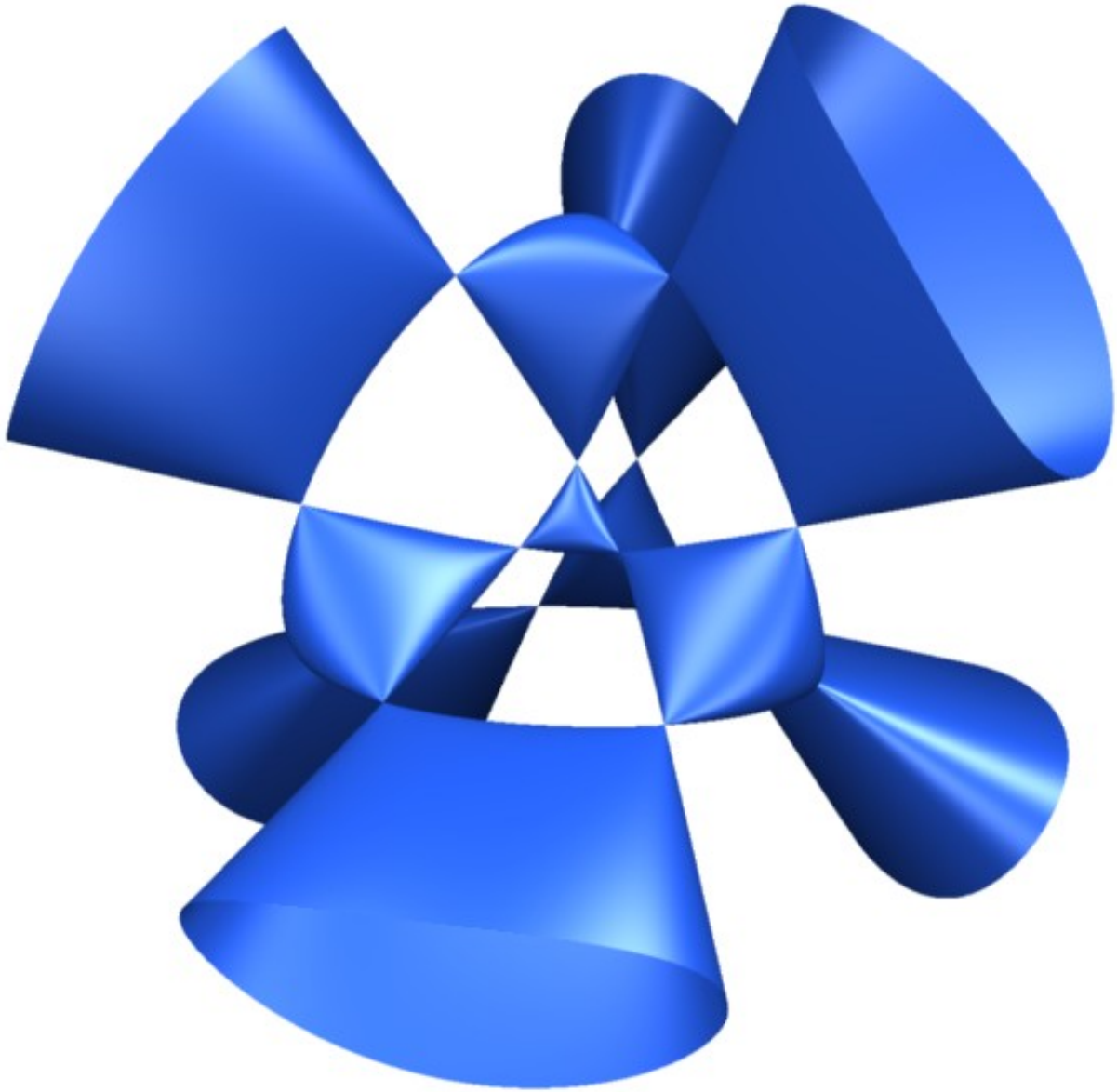
Alice computes $(a, \pm P) \mapsto \pm[a]P$; Bob $(b, \pm[a]P) \mapsto \pm[ab]P$...

Elliptic curves: work on $x$-line $\mathbb{P}^1 = \mathcal{E}/\langle \pm 1 \rangle$

*Eg. Curve25519 (Bernstein 2006).*

Genus 2: work on the Kummer surface $\mathcal{K}_\mathcal{C} := \mathcal{J}_\mathcal{C}/\langle \pm 1 \rangle$.

# What a Kummer surface looks like



# *Moving from $\mathcal{J}_\mathcal{C}$ to the Kummer $\mathcal{K}_\mathcal{C}$*

Quotient map $\mathbf{x} : \mathcal{J}_\mathcal{C} \longrightarrow \mathcal{K}_\mathcal{C}$ (ie $\mathbf{x}(P) = \pm P$)

No group law on $\mathcal{K}_\mathcal{C}$: $\mathbf{x}(P)$ and $\mathbf{x}(Q)$ determines $\mathbf{x}(P \oplus Q)$ and $\mathbf{x}(P \ominus Q)$, but we can't tell which is which.

Still, $\ominus[m](P) = [m](\ominus P)$ for any $m \in \mathbb{Z}$ and $P \in \mathcal{J}_\mathcal{C}$, so we do have a "scalar multiplication" on $\mathcal{K}_\mathcal{C}$:

$$[m] : \mathbf{x}(P) \longmapsto \mathbf{x}([m]P) \ .$$

*Problem:* How do we compute $[m]$ efficiently, *without* $\oplus$?

Any 3 of $\mathbf{x}(P)$, $\mathbf{x}(Q)$, $\mathbf{x}(P \ominus Q)$, and $\mathbf{x}(P \oplus Q)$ determines the 4th, so we can define

*pseudo-addition*

$$\texttt{xADD} : (\mathbf{x}(P), \mathbf{x}(Q), \mathbf{x}(P \ominus Q)) \longmapsto \mathbf{x}(P \oplus Q)$$

*pseudo-doubling*

$$\texttt{xDBL} : \mathbf{x}(P) \longmapsto \mathbf{x}([2]P)$$

Bonus: easier to hide/avoid special cases in xADD than $\oplus$.

# Kummer surface arithmetic

Let $0_K$ be the image of $0_{\mathcal{J}_\mathcal{C}}$ in $\mathcal{K}_\mathcal{C}$, and define

$$\mathcal{M} : ((x_1 : y_1 : z_1 : t_1), (x_2 : y_2 : z_2 : t_2)) \longmapsto (x_1x_2 : y_1y_2 : z_1z_2 : t_1t_2)$$
$$\mathcal{S} : (x : y : z : t) \longmapsto (x^2 : y^2 : z^2 : t^2)$$
$$\mathcal{I} : (x : y : z : t) \longmapsto (1/x : 1/y : 1/z : 1/t)$$
$$\mathcal{H} : (x : y : z : t) \longmapsto (x' : y' : z' : t') \text{ where } \begin{cases} x' = x + y + z + t\ , \\ y' = x + y - z - t\ , \\ z' = x - y + z - t\ , \\ t' = x - y - z + t\ . \end{cases}$$

- $\texttt{xDBL}(\mathbf{x}(P)) = \mathcal{M}(\mathcal{HM}(\mathcal{S}(\mathcal{HS}(\mathbf{x}(P)))), \mathcal{IH}(0_K)), \mathcal{I}(0_K))$
- $\texttt{xADD}(\mathbf{x}(P), \mathbf{x}(Q), \mathbf{x}(P \ominus Q))$
  $= \mathcal{M}(\mathcal{HM}(\mathcal{M}(\mathcal{HS}(\mathbf{x}(P)), \mathcal{HS}(\mathbf{x}(Q)))), \mathcal{IH}(0_K)), \mathcal{I}(\mathbf{x}(P \ominus Q)))$

  *The green things here are constants, or constants in practice.*

Evaluate $[m]$ by combining xADDs and xDBLs
using differential addition chains
*(ie. every $\oplus$ has summands with known difference).*
Classic example: the Montgomery ladder.

**Algorithm 3** The Montgomery ladder in a group

1: **function** $\textsc{Ladder}(m = \sum_{i=0}^{\beta-1} m_i 2^i,\ P)$
2: $(R_0, R_1) \leftarrow (0, P)$
3: **for** $i := \beta - 1$ down to 0 **do**
4: $(R_{m_i}, R_{\neg m_i}) \leftarrow ([2]R_{m_i}, R_{m_i} \oplus R_{\neg m_i})$
5: **end for** ▷ invariant: $(R_0, R_1) = ([\lfloor m/2^i \rfloor]P, [\lfloor m/2^i \rfloor + 1]P)$
6: **return** $R_0$ ▷ $R_0 = [m]P$, $R_1 = [m]P \oplus P$
7: **end function**

For each group operation $R_0 \oplus R_1$, the difference $R_0 \ominus R_1$ is *fixed*
$\implies$ trivial adaptation from $\mathcal{J}_{\mathcal{C}}$ to $\mathcal{K}_{\mathcal{C}}$

**Algorithm 4** The Montgomery ladder on the Kummer

1: **function** $\textsc{Ladder}(m = \sum_{i=0}^{\beta-1} m_i 2^i, \mathbf{x}(P))$
2: $(x_0, x_1) \leftarrow (\mathbf{x}(0), \mathbf{x}(P))$
3: **for** $i := \beta - 1$ down to 0 **do**
4: $(x_{m_i}, x_{\neg m_i}) \leftarrow (\mathtt{xDBL}(x_{m_i}), \mathtt{xADD}(x_0, x_1, \mathbf{x}(P))$
5: **end for** ▷ invariant: $x_0 = \mathbf{x}([\lfloor m/2^i \rfloor]P, x_1 = \mathbf{x}([\lfloor m/2^i \rfloor + 1]P)$
6: **return** $x_0$ $(= \mathbf{x}([m]P))$
7: **end function**

High symmetry of $\mathcal{K}_{\mathcal{C}}$
$\Longrightarrow$ fast, vectorizable xADD and xDBL (Gaudry)

$\Longrightarrow$ very fast Kummer-based Diffie–Hellman implementations
Eg. Bos–Costello–Hisil–Lauter (2013/14),
Bernstein–Chuengsatiansup–Lange–Schwabe (Asiacrypt 2014).

# *Pulling a y-rabbit out of an x-hat*

Kummer multiplication computes $\mathbf{x}([m]P)$ from $\mathbf{x}(P)$
—but we need $[m]P$ for signatures...

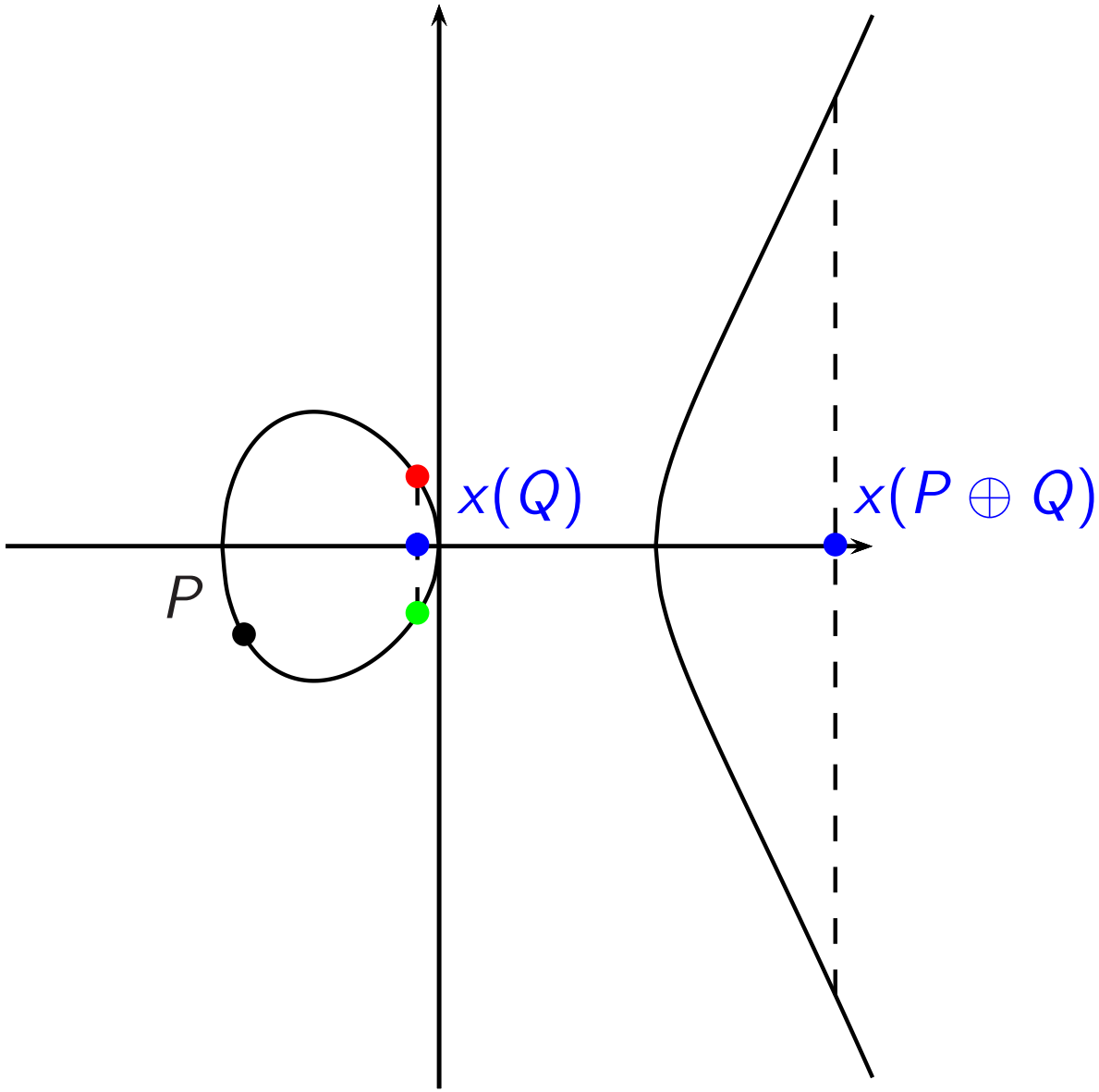
Mathematically, we threw away the sign:
you can't deduce $[m]P$ from $P$ and $\mathbf{x}([m]P)$.

But there's a trick: if you computed $\mathbf{x}([m]P)$
using the Montgomery ladder, then you can!

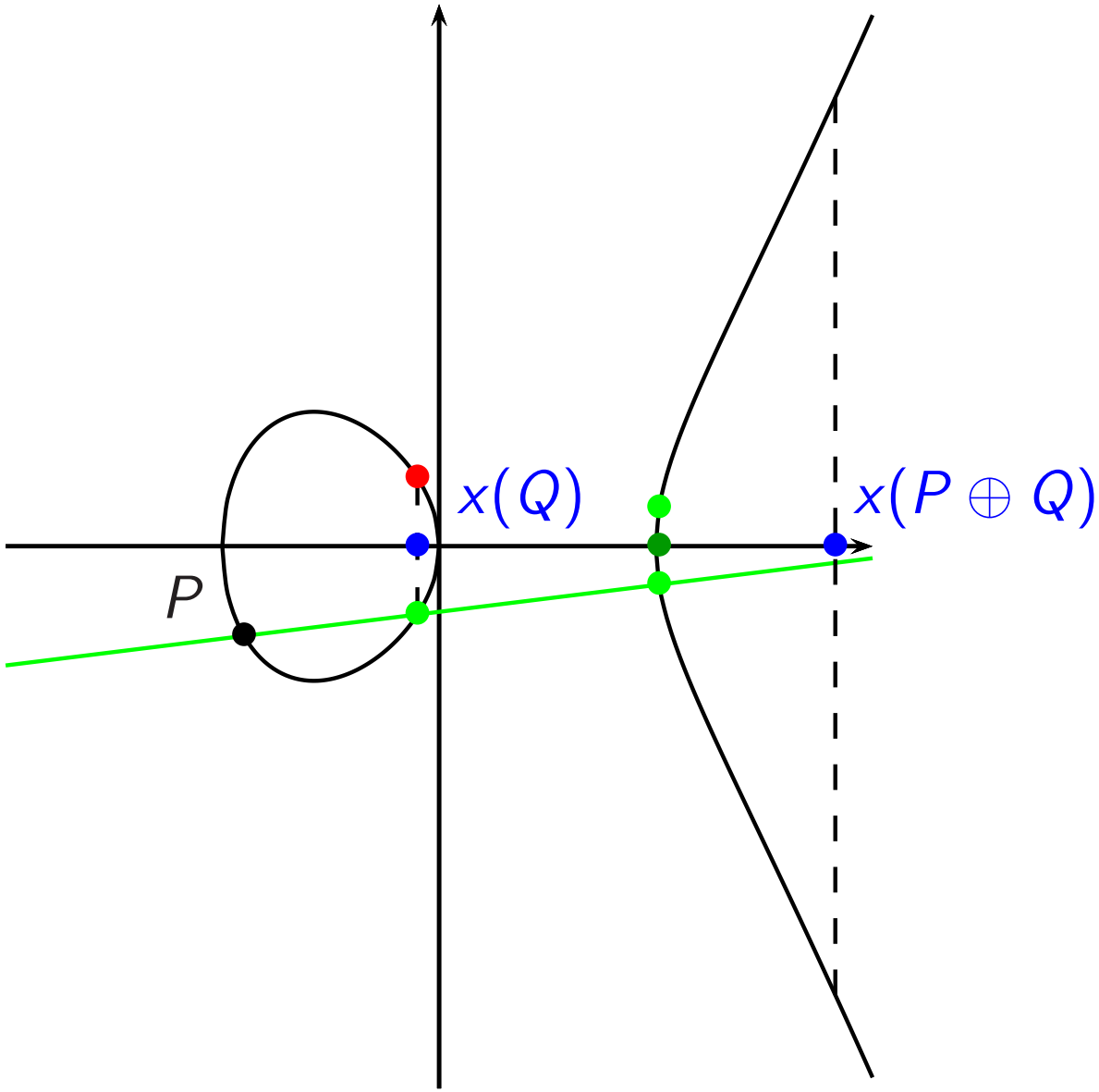
At the end of the loop, $x_0 = \mathbf{x}([m]P)$ and $x_1 = \mathbf{x}([m]P \oplus P)$;
and $P$, $\mathbf{x}(Q)$, and $\mathbf{x}(Q \oplus P)$ uniquely determines $Q$ (for any $Q$).

This is an old trick for elliptic curves: cf. López–Dahab (CHES 99), Okeya–Sakurai (CHES 01), Brier–Joye (PKC 02).

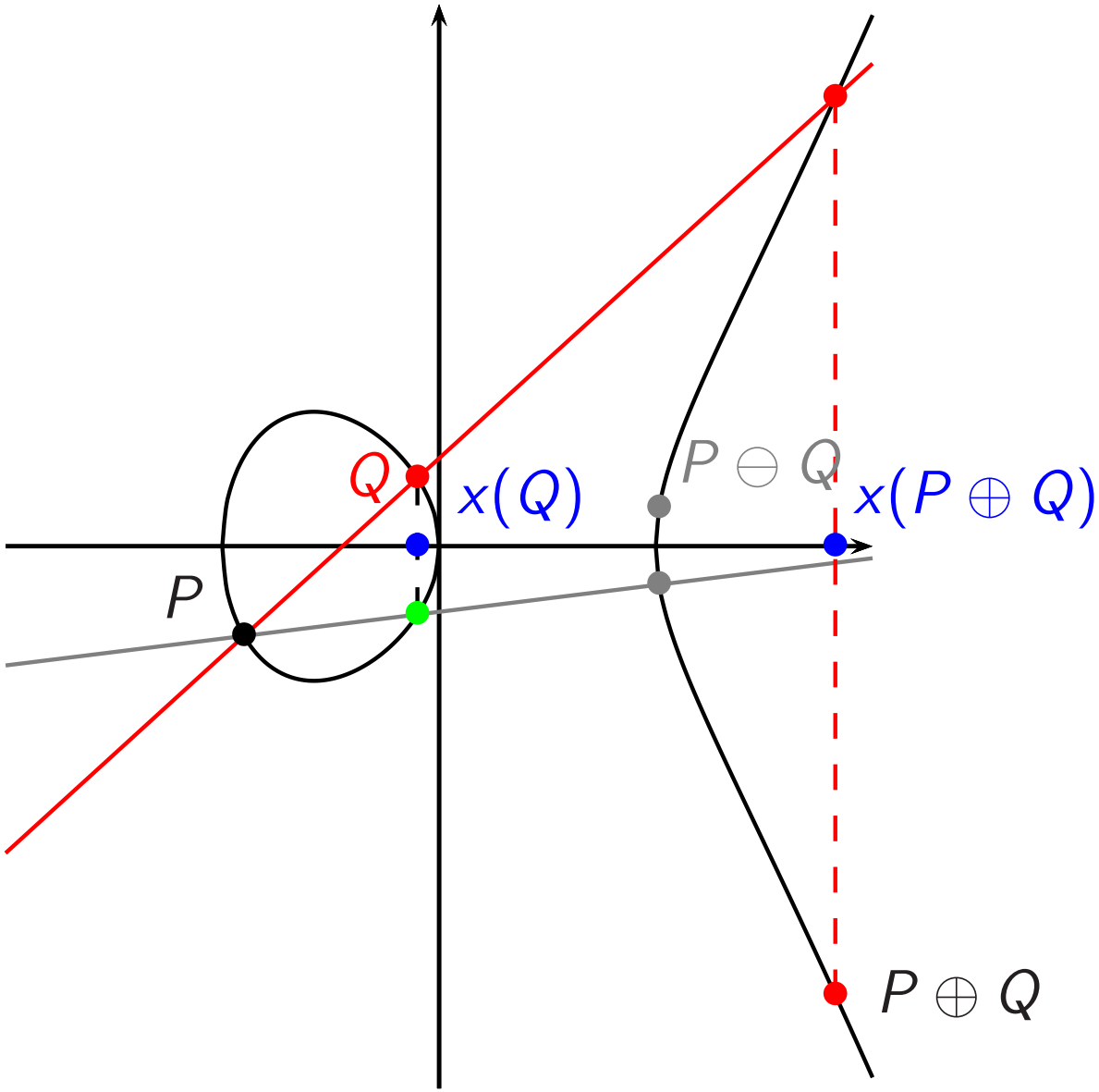
## *Given $P$, $\mathbf{x}(Q)$, and $\mathbf{x}(P \oplus Q)$. Is $Q =$ ● or ●?*



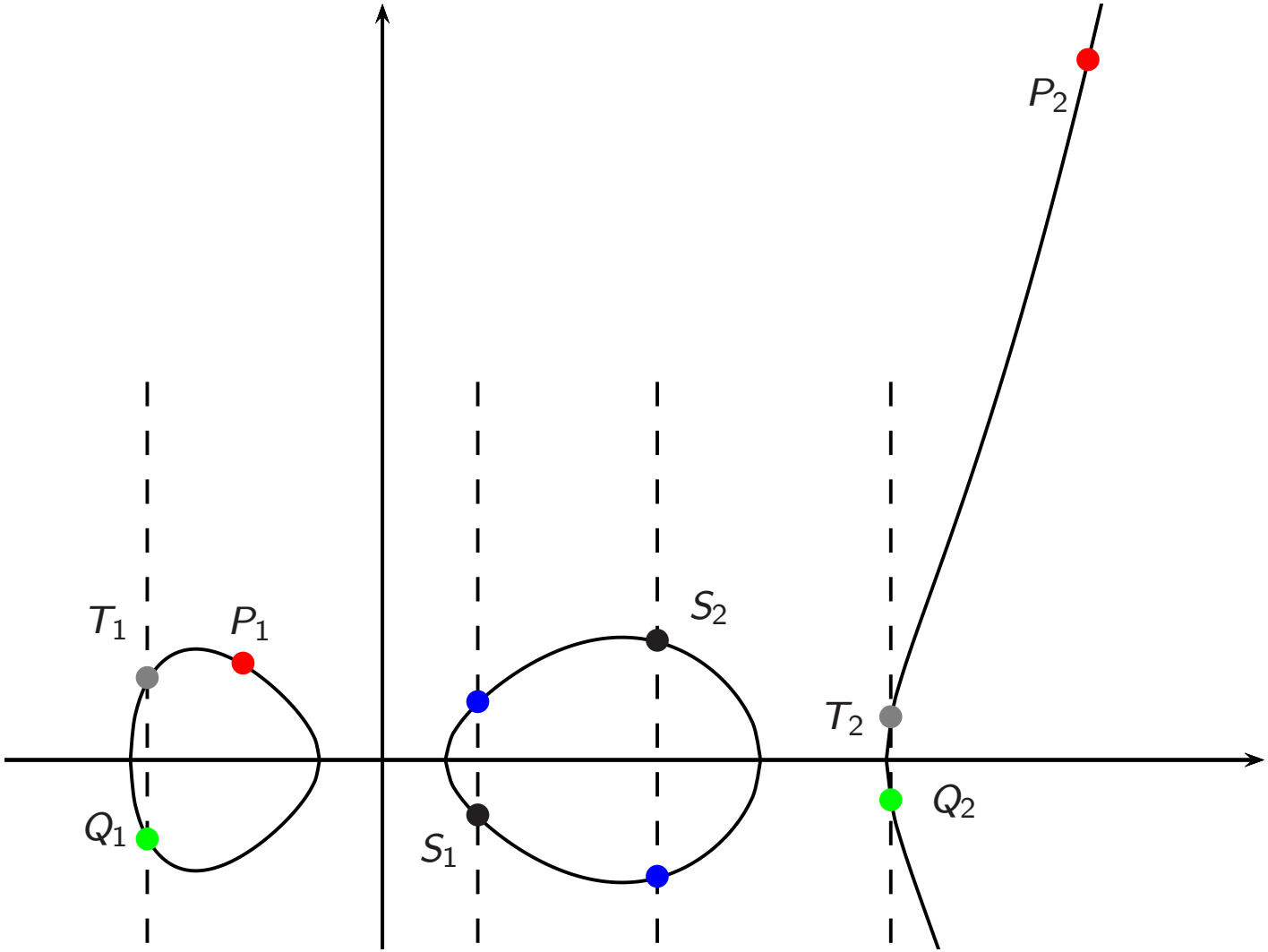
## *Line through $P$ and $\bullet$: "sum" incompatible with $\mathbf{x}(P \oplus Q)$.*



# *Line through $P$ and $\bullet$: "sum" fits with $\mathbf{x}(P \oplus Q)$, so $Q = \bullet$*

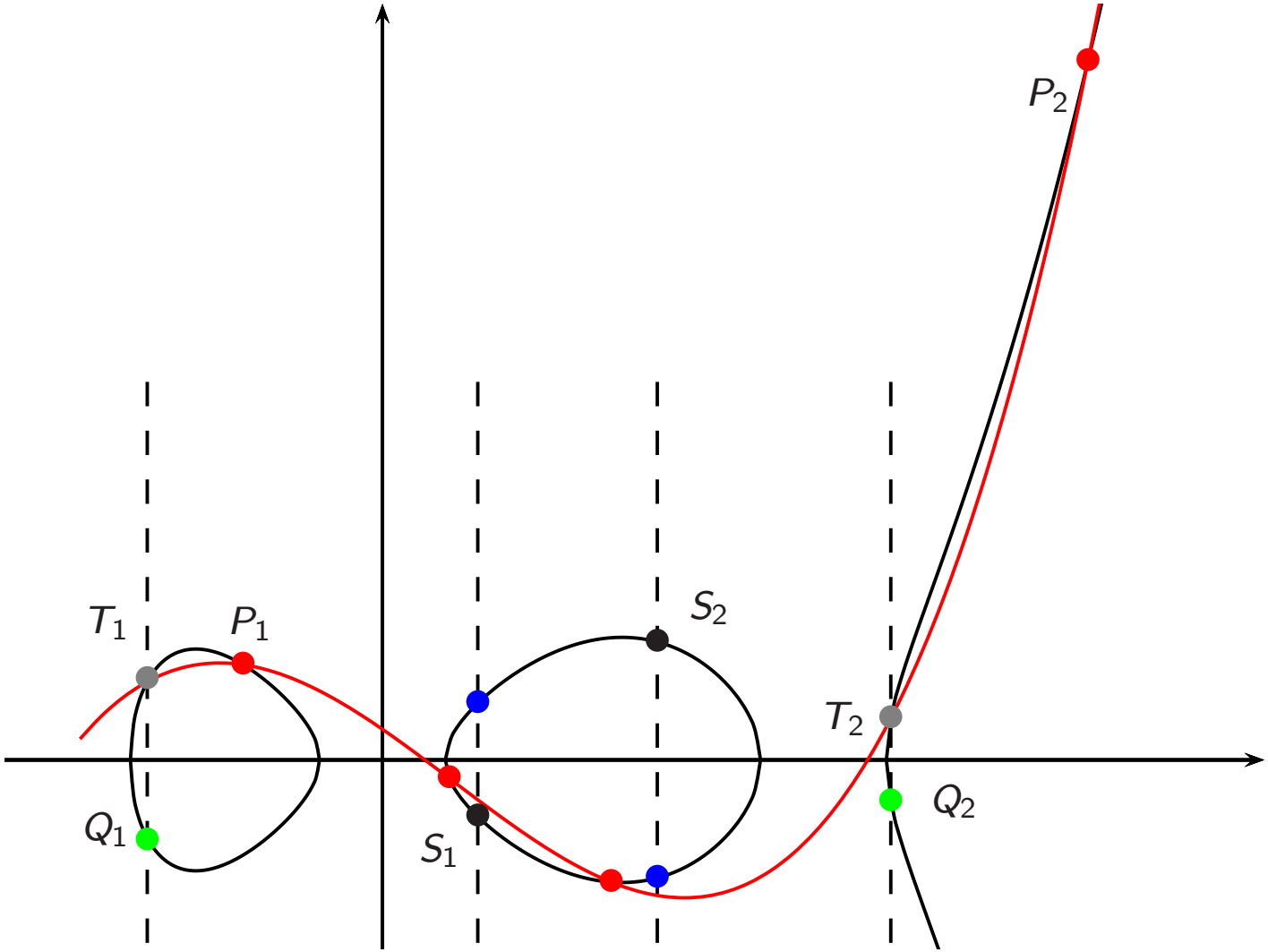


# Genus 2 group law: $\{P_1, P_2\} \oplus \{Q_1, Q_2\} = \{S_1, S_2\}$



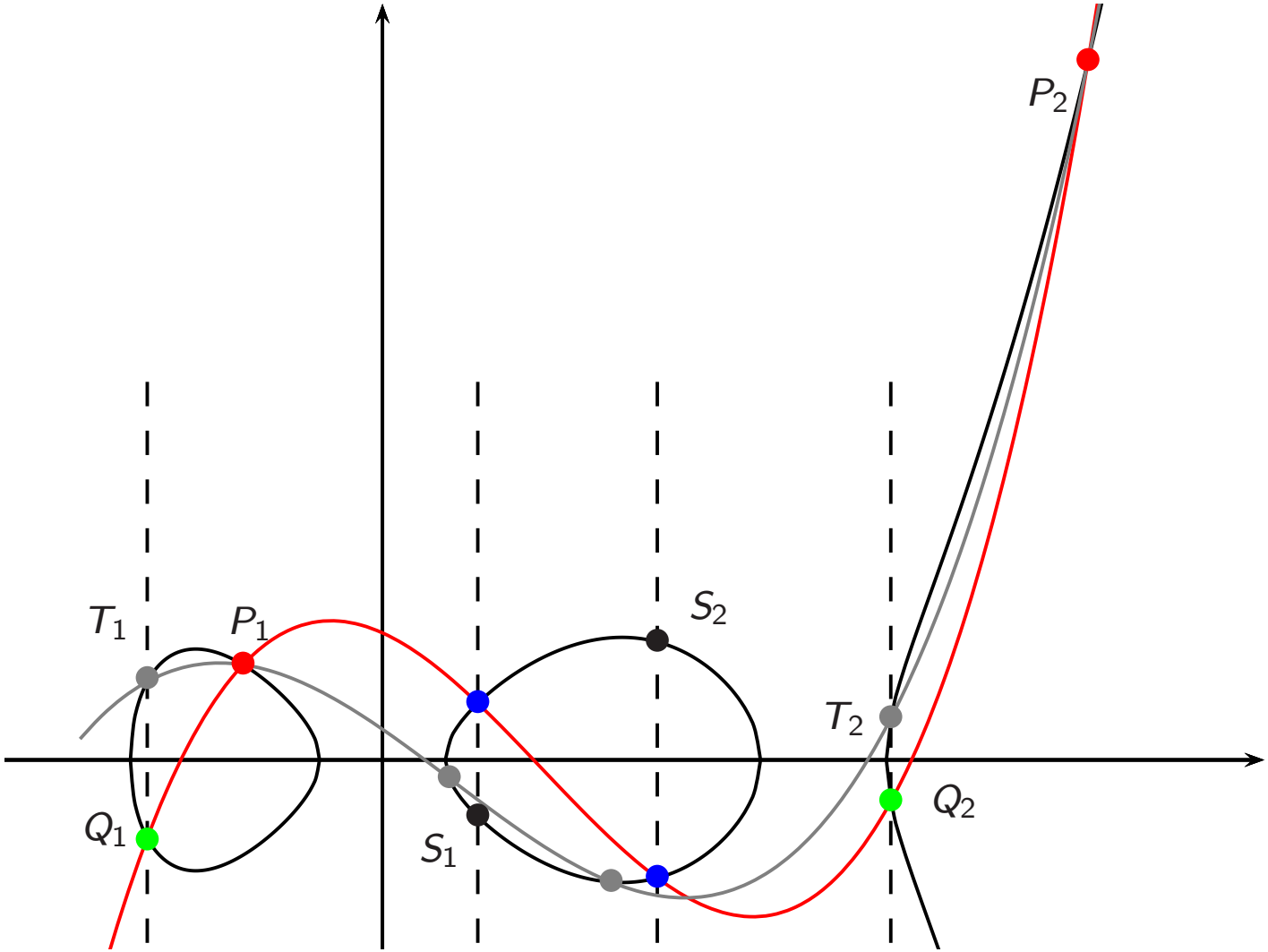

We know $\{P_1, P_2\}$, $\mathbf{x}(\{Q_1, Q_2\}) = \mathbf{x}(\{T_1, T_2\})$, and $\mathbf{x}(\{S_1, S_2\}) = \mathbf{x}(\{R_1, R_2\})$.
How can we find $\{Q_1, Q_2\}$ (and not $\{T_1, T_2\}$)?

# Genus 2 group law: $\{P_1, P_2\} \oplus \{Q_1, Q_2\} = \{S_1, S_2\}$




Choosing $\{T_1, T_2\}$ as (the wrong) preimage of $\mathbf{x}(\{Q_1, Q_2\})$ yields a cubic incompatible with $\mathbf{x}(\{S_1, S_2\})$.

# Genus 2 group law: $\{P_1, P_2\} \oplus \{Q_1, Q_2\} = \{S_1, S_2\}$




There is only one cubic compatible with $\{P_1, P_2\}$, $\mathbf{x}(\{Q_1, Q_2\})$, and $\mathbf{x}(\{S_1, S_2\})$.
It then intersects with $\mathcal{C}$ in the (correct) preimage $\{Q_1, Q_2\}$.

## *Chung–Costello–S. (SAC 2016):*

`Recover` algorithm implementing this trick for genus 2.

*Input*: $P$ in $\mathcal{J}_{\mathcal{C}}(\mathbb{F}_q)$, $\mathbf{x}(Q)$ and $\mathbf{x}(P \oplus Q)$ in $\mathcal{K}_{\mathcal{C}}(\mathbb{F}_q)$ (for some unknown $Q$ in $\mathcal{J}_{\mathcal{C}}(\mathbb{F}_q)$).

*Output*: $Q$ in $\mathcal{J}_{\mathcal{C}}(\mathbb{F}_q)$.

Application: put
$(P, \mathbf{x}(Q), \mathbf{x}(P \oplus Q)) = (P, \mathbf{x}([m]P), \mathbf{x}([m+1]P))$
(ie, $P$ with output of $\mathbf{x}$-Montgomery ladder); then

$$\texttt{Recover}(P, \mathbf{x}([m]P), \mathbf{x}([m+1]P)) = [m]P \ .$$

$\Longrightarrow$ 1D (Montgomery) and 2D (Bernstein) SM.

# *Chung–Costello–S. (SAC 2016):*

**Algorithm 5** Montgomery/Kummer-based multiplication on the Jacobian

**function** $\textsc{ScalarMultiply}(m = \sum_{i=0}^{\beta-1} m_i 2^i,\ P)$
  $(x_0, x_1) \leftarrow (\mathbf{x}(0), \mathbf{x}(P))$
  **for** $i := \beta - 1$ down to 0 **do** ▷ Montgomery ladder
    $(x_{m_i}, x_{\neg m_i}) \leftarrow (\mathtt{xDBL}(x_{m_i}), \mathtt{xADD}(x_0, x_1, \mathbf{x}(P))$
  **end for** ▷ invariant: $x_0 = \mathbf{x}([\lfloor m/2^i \rfloor]P)$, $x_1 = \mathbf{x}([\lfloor m/2^i \rfloor + 1]P)$
  $Q \leftarrow \mathtt{Recover}(P, x_0, x_1)$ ▷ $Q = [m]P$
  **return** $Q$
**end function**

Now: your fast Kummer implementations can be easily bootstrapped to full Jacobian group implementations; fast Diffie–Hellman code now yields efficient signatures.

# $\mu$Kummer

## Kummer crypto for microcontrollers

*Renes–Schwabe–S.–Batina, CHES 2016*

*A fiendishly simple plan:*

1. Build efficient Diffie–Hellman software for small devices, based on Kummer arithmetic; Done
2. Bootstrap it to a full Schnorr signature scheme, using the previous trick; Done
3. Take over the world. (In progress)

# $\mu$Kummer software

Free, public domain C/assembly implementation of Diffie–Hellman key exchange and Schnorr signatures targeting the 128-bit security level on

AVR ATMega (8-bit architecture)

ARM Cortex M0 (32-bit architecture)

Download it: `http://www.cs.ru.nl/~jrenes`

## Comparison for 8-bit architecture (AVR ATmega):

| Protocol | Object | kCycles | Stack bytes |
|---|---|---|---|
| Diffie–Hellman | Curve25519 | 13900 | 494 |
| | $\mu$**Kummer** | 9513 (68%) | 99 (20%) |
| Schnorr signing | Ed25519 | 19048 | 1473 |
| | $\mu$**Kummer** | 10404 (55%) | 926 (63%) |
| Schnorr verifying | Ed25519 | 30777 | 1226 |
| | $\mu$**Kummer** | 16241 (53%) | 992 (75%) |

## Comparison for 32-bit architecture (ARM Cortex M0):

| Multiplication for | Object | kCycles | Stack bytes |
|---|---|---|---|
| Diffie–Hellman | Curve25519 | 3590 | 548 |
| | $\mu$**Kummer** | 2634 (73%) | 248 (45%) |
| Schnorr | NIST-P256 | 10730 | 540 |
| | $\mu$**Kummer** | 2709 (25%) | 968 (179%) |

*Curve25519 = Düll-Haase-Hinterwälder-Hutter-Paar-Sánchez-Schwabe '15*
*Ed25519 = Nascimento-López-Dahab '15*
*NIST-P256 = Wenger-Unterluggauer-Werner '13*

Recovery:

http://eprint.iacr.org/2016/777

$\mu$Kummer:

http://eprint.iacr.org/2016/366

Download the software:

http://www.cs.ru.nl/~jrenes