Discrete logarithms in cryptographically interesting characteristic-three finite fields

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  - solved the DLP in the order-r subgroup of $\mathbb{F}^*_{3^{6 \cdot 509}}$ in 223 CPU years;
  - expect to solve the DLP in the order-r subgroup of $\mathbb{F}^*_{3^{6 \cdot 1429}}$ within 173 days on clusters of 9000 and 1500 cores, thanks to Guillevic's new descent.

Pairing-Based Cryptography

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- Immediate property: for any integer k,

$$\hat{e}(kQ, R) = \hat{e}(Q, R)^k = \hat{e}(Q, kR).$$

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- **Short digital signatures**
  - Boneh–Lynn–Shacham, 2001.
  - Zang–Safavi-Naini–Susilo, 2004.

Small-Characteristic Pairings

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Most common pairing maps:

- Weil pairings.
- Tate pairings and modifications (Eta, Ate, ...).

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- The $k = 6$ pairings derived from supersingular elliptic curves over $\mathbb{F}_{3^n}$:
  - $Y^2 = X^3 - X + 1$; and
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- The $k = 12$ pairing derived from supersingular gen.-2 curves over $\mathbb{F}_{2^n}$:
  - $Y^2 + Y = X^5 + X^3$; and
  - $Y^2 + Y = X^5 + X^3 + 1$.

Security of Small-Characteristic Pairings (Prior to 2013)

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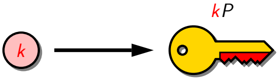

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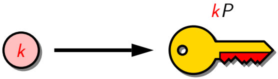

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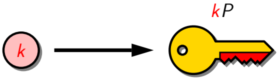

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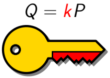

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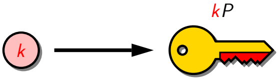

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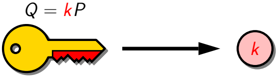

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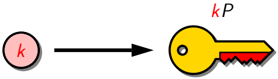

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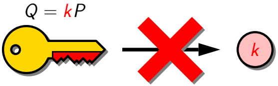

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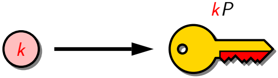

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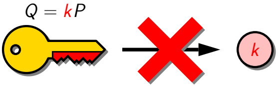

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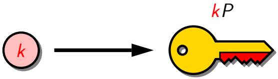

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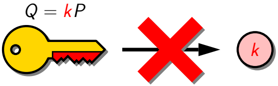

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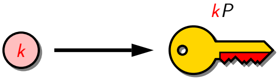

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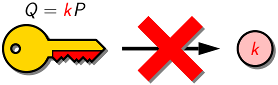

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DLP: given $h \in \mathbb{G}_T$, find $0 \leq i < r$ such that $h = g^i$.

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- For pairing-based cryptography over supersingular curves:
  - The embedding degree is relatively small ($k = 4$, 6, or 12).
  - So, the finite field $\mathbb{F}_{q^k}$ (containing $\mathbb{G}_T$) is not very large.

DLP algorithms for small-characteristic fields $\mathbb{F}_Q$

- Subexponential running time, for $0 < \alpha < 1$ and $c > 0$, at input Q:

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Table: Security of small-characteristic parings as in 2012 (DLP in $\mathbb{F}_{p^{kn}}$)

Underlying field ($\mathbb{F}p^n$)	$\mathbb{F}_{2^n}$	$\mathbb{F}_{3^n}$	$\mathbb{F}_{2^n}$
Embedding degree (k)	4	6	12
Lower security ($\approx 2^{64}$)	$n = 239$	$n = 97$	$n = 127$
Medium security ($\approx 2^{80}$)	$n = 373$	$n = 163$	$n = 163$
Standard security ($\approx 2^{128}$)	$n = 1223$	$n = 509$	$n = 367$
Higher security ($\approx 2^{192}$)	$n = 3041$	$n = 1429$	$n \approx 983$

Joux-Lercier algorithm for $\mathbb{F}_Q = \mathbb{F}_{q^n}$

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- Later in 2012, Joux [Joux12] introduced a technique that improved the [JL06] algorithm to $L_Q[\frac{1}{3}, 0.961]$.

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Early Contributions

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- The $\mathbb{F}_{3^{6\cdot 137}}$ computation was the first to illustrate the effectiveness of the new algorithms in a cryptographically-interesting field.

More Improvements

Practical improvements

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The 509's Computations

Latest computations on $\mathbb{F}_{3^{6 \cdot 509}}$

July 18 2016, A.-Canales-Cruz-Menezes-Oliveira-Rivera-Rodríguez

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- Let $E : y^2 = x^3 - x + 1$ be the supersingular elliptic curve over $\mathbb{F}_{3^{509}}$ with $|E(\mathbb{F}_{3^{509}})| = 7r$, where $r = (3^{509} - 3^{255} + 1)/7$ is a 804-bit prime.

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DLP algorithm	Copp.04	JL06	Joux12	Joux13-QPA13	JP14-GKZ14
Run time	2^{128}	2^{111}	2^{103}	2^{75}	2^{49}

Running-time

Computation stage	CPU time (years)		CPU frequency (GHz)
Finding logarithms of quadratic polynomials			
Relation generation	0.01	(CS Dept.)	3.20
Linear algebra	0.50	(CS Dept.)	2.40
Finding logarithms of cubic polynomials			
Relation generation	0.15	(CS Dept.)	3.20
Linear algebra	43.88	(ABACUS)	2.60
Finding logarithms of quartic polynomials			
Relation generation	4.07	(CS Dept.)	2.60
Linear algebra	96.02	(ABACUS)	2.60
Descent			
Continued-fractions (254 to 40)	51.71	(CS Dept.)	2.87
Classical (40 to 21)	9.99	(CS Dept., U Wat.)	2.66
Classical (21 to 15)	10.24	(CS Dept., U Wat.)	2.66
Gröbner bases (15 to 4)	6.27	(CS Dept., U Wat.)	3.00
Total CPU time (years)	222.81		

Table: CPU times of each stage of the discrete logarithm computation in $\mathbb{F}_{3^{6\cdot 509}}$.

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  - Logarithms in the VM are read by Magma and those in the HD from some C-codes called from Magma. The System V shared memory is used for the Magma-C for the interprocess communication.

The Guillevic Descent

Guillevic's descent method (July 2016)

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- In our case, we choose n' so that $3^{6n'-3n} \gg q^{n'}/N_q(m, n')$, where $N_q(m, n')$ denotes the number of monic m-smooth degree-n' polynomials in $\mathbb{F}_q[X]$.

Discrete logarithms in $\mathbb{F}_{3^{6\cdot 709}}$

- Joux's Frobenius representation of $\mathbb{F}_{3^{6\cdot n}}$ requires a degree-n irreducible factor of $h_1(X)X^q - h_0(X)$ over $\mathbb{F}_{3^6}$, where $\max(\deg h_0, \deg h_1) = 2$.

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- In contrast, the expected cost of the continued-fractions descent to express h as the ratio of two 40-smooth degree-354 polynomials is $2^{81.3} M_q$.
- Since the rest of the descent is similar to what we did for $\mathbb{F}_{3^{6\cdot509}}$, we conclude that the logarithm x can be computed with the same effort as in $\mathbb{F}_{3^{6\cdot509}}$.

Discrete Logarithms at The 192-bit Security Level

Discrete logarithms in $\mathbb{F}_{3^{6\cdot 1429}}$

- $E : Y^2 = X^3 - X - 1$ a supersingular elliptic curve over $\mathbb{F}_3$. $|E(\mathbb{F}_{3^{1429}})| = cr$, where $c = 7622150170693$ and $r = (3^{1429} - 3^{715} + 1)/c$, a 2223-bit prime.

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Finding logarithms of quadratic polynomials	
Degree 1 and 2	$2^{50.7}$
Degree 3	$2^{56.9}$
Degree 4 (36/728)	$2^{56.3}$
Descent	
Guillevic (1428 to 71)	$2^{62.4}$
Classical (71 to 32)	$2^{61.8}$
Classical (31 to $\{1, \ldots, 16, 18, 20, 22, 24, 28, 32\}$)	$2^{59.2}$
Small degree ($\{5, \ldots, 16, 18, 20, 22, 24, 28, 32\}$ to 4)	$2^{60.0}$
Total cost	$2^{63.4}$

Table: Estimated costs of the main steps for computing discrete logarithms in $\mathbb{F}_{3^{6\cdot 1429}}$.

Feasibility of computing discrete logarithms in $\mathbb{F}_{3^{6\cdot 1429}}$

- We assume that we have access to a 9000-core cluster $\mathcal{A}$, where each core has access to 16 gigabytes of shared RAM, such as ABACUS-Cinvestav.

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Computation	Cluster	# cores	# days
Degree-3	$\mathcal{A}$	5824	2
Degree-4	$\mathcal{A}$	9000	1
Guillevic descent	$\mathcal{A}$	9000	59
First classical descent	$\mathcal{A}$	9000	39
Second classical descent	$\mathcal{A}$	9000	7
Small degree descent	$\mathcal{B}$	1500	65
Total time			173

Table: Estimated calendar time for computing discrete logarithms in $\mathbb{F}_{3^{6\cdot 1429}}$ using clusters $\mathcal{A}$ and $\mathcal{B}$.

Open problem

Since the effort in the previous slide is still beyond the reach of the computer resources available to us, it would be worthwhile to improve the descent strategy.

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Thanks For Your Attention!

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